

Corrected Loop Vertex Expansion for Φ_2^4 Theory

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Abstract

This paper is an extended erratum to [1], in which the classic construction and Borel summability of the ϕ_2^4 Euclidean quantum field theory was revisited combining a multi-scale analysis with the constructive method called Loop Vertex Expansion (LVE). Unfortunately we discovered an important error in the method of [1]. We explain the mistake, and provide a new, correct construction of the ϕ_2^4 theory according to the LVE.

1 Introduction

The Loop Vertex Expansion (LVE) [2, 3, 4] is a constructive field theory technique which combines a forest formula [5, 6], a replica trick and the intermediate field representation. It can compute the connected functions of stable Bosonic quantum field theories with both infrared and ultraviolet cutoffs and a small-coupling without introducing the space-time lattices and cluster expansions of standard constructive theory [7, 8, 9]. It can also be considered as a convergent reshuffling of Feynman graphs using canonical combinatorial tools [10].

To remove cutoffs in models with ultraviolet divergences requires to extend the LVE technique to allow for renormalization, that is for the explicit cancellation of counterterms. A first method, called “cleaning expansion” has been introduced to cure the ultraviolet divergences of superrenormalizable models such as the commutative ϕ_2^4 model [1] and the 2-dimensional Grosse-Wulkenhaar model [11].

We recently discovered an important error in this method and in particular in the bounds of [1]. Unfortunately it does not seem possible to fix the cleaning expansion itself. Since it expands all loop vertices up to infinity, an arbitrary number of them can require renormalization, and the expansion fails to converge. The attentive reader can trace back the mistake to the section 5 of [1], where some crucial sums over the scales of the tadpoles were simply forgotten.

In this paper we correct this mistake, replacing the “cleaning expansion” with a more careful expansion, which keeps hardcore constraints between the exponential of the interaction in different slices. This multiscale loop vertex expansion has been defined and checked to work in detail for a simpler toy model in [12]. Here we adapt it to the ϕ_2^4 theory, restricting ourselves for simplicity to remove the ultraviolet cutoff only. Hence we keep the theory in a single unit square.

Of course the construction of ϕ_2^4 [7, 13, 14] and its Borel summability [15] are classic milestones of constructive theory. The purpose of this paper is just to obtain these results again with a multislice LVE. We prove analyticity in the coupling constant of the free energy¹ of the theory in a cardioid-shaped domain (see Figure 8) which has opening angle arbitrarily close to 2π . This domain, which is the natural one for LVE-type expansions [16], is larger than the ones usually considered in the constructive literature [9, 15], which are of Nevanlinna-Sokal or Watson-type, hence have opening angles respectively π or $\pi + \epsilon$.

It is also our hope and goal to adapt ultimately the MLVE techniques to treat just-renormalizable models. An important motivation for this is the constructive treatment of quantum field theories with non-local

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¹Extensions to arbitrary Schwinger functions are left as an exercise to the reader.

interactions such as the four dimensional Grosse-Wulkenhaar non-commutative field theory [17] or tensor group field theories [18, 19], for which the LVE seems clearly better adapted than traditional constructive techniques [16, 20, 21].

We refer to [22, 23, 24, 25, 26] for a recent extensive reformulation of the constructive renormalization group multiscale techniques relying on scaled lattices.

2 The Model and its Slice-Testing Expansion

We consider the free Bosonic ϕ^4_2 theory in a fixed volume, namely the unit square $[0, 1]^2$ with coupling constant λ . From now on any spatial integral has to be understood as restricted to $[0, 1]^2$ and we use the notation Tr to mean $\int_{[0, 1]^2} d^2x$. The formal partition function of the theory with source J is

$$Z(J, \lambda) = \int d\mu_C(\phi) e^{\text{Tr } J(x)\phi(x) - \frac{\lambda}{2} \text{Tr } \phi^4(x)} \quad (2.1)$$

where $d\mu_C$ is the normalized Gaussian measure with covariance or propagator

$$C(x, y) = \frac{1}{4\pi} \int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}}, \quad (2.2)$$

with x and y restricted to $[0, 1]^2$, hence with free boundary conditions; the theory could equally well be considered on the two dimensional torus with periodic boundary conditions without significant change in the analysis. A main problem in quantum field theory is to compute the logarithm $\log Z(J, \lambda)$ which is the generating function of the connected Schwinger functions.

The covariance at coinciding points $C(x, x) = T$ corresponds to a self-loop or tadpole in perturbation theory. It diverges logarithmically in the ultraviolet cutoff and is the only primitive ultraviolet divergence of the theory. Renormalization reduces to Wick ordering, hence the renormalized model has partition function:

$$\begin{aligned} Z(J, \lambda) &= \int d\mu_C(\phi) e^{\text{Tr } J(x)\phi(x) - \frac{\lambda}{2} \text{Tr } :\phi^4(x):} = \int d\mu_C e^{\text{Tr } J(x)\phi(x) - \frac{\lambda}{2} \text{Tr } [\phi^4 - 6T\phi^2 + 3T^2]} \\ &= \int d\mu_C e^{\text{Tr } J(x)\phi(x) - \frac{\lambda}{2} \text{Tr } [(\phi^2 - 3T)^2 - 6T^2]}, \end{aligned} \quad (2.3)$$

where the Wick ordering in $:\phi^4(x): \equiv \phi^4 - 6T\phi^2 + 3T^2$ is taken with respect to C [14].

These expressions are formal and to define the theory one needs to introduce an ultraviolet cutoff. This is most conveniently done in a multiscale representation [9] which slices the propagator in the parametric representation, then keeps only a finite number of slices.

2.1 Slices and Intermediate Field Representation

We fix an integer $M > 1$, and we slice the propagator as usual in the multislice analysis [12, 9], defining the ultraviolet cutoff as a maximal slice index j_{max} . The ultraviolet limit corresponds to $j_{max} \rightarrow \infty$. Hence we define² $C = C_{\leq j_{max}} = \sum_{j=0}^{j_{max}} C_j$ with:

$$C_0(x, y) = \int_1^\infty e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}} \frac{d\alpha}{\alpha} \leq K e^{-c|x-y|}, \quad (2.4)$$

$$C_j(x, y) = \int_{M^{-2j}}^{M^{-2(j-1)}} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}} \frac{d\alpha}{\alpha} \leq O(1) e^{-O(1)M^j|x-y|} \text{ for } j \geq 1. \quad (2.5)$$

Throughout this paper we use the time-honored constructive convention of noting $O(1)$ any inessential constant. The set of slice indices $\mathcal{S} = [0, j_{max}]$ has $1 + j_{max}$ elements.

We also put $T_j = C_j(x, x) \leq O(1)$, and

$$C_{\leq j} = \sum_{k=0}^j C_k = \int_{M^{-2j}}^\infty e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}} \frac{d\alpha}{\alpha}, \quad T_{\leq j} = \sum_{k=0}^j T_k, \quad (2.6)$$

²Beware we choose the convention of *lower* indices for slices, as in [12], not upper indices as in [9].

The partition function $Z^{j_{max}}(J, \lambda)$ obtained by substituting $C_{\leq j_{max}}$ instead of C and $T_{\leq j_{max}}$ instead of T in (2.3) is now well defined. For simplicity we put from now on the source $J = 0$, defining $Z(\lambda) = Z(J, \lambda)|_{J=0}$. The ultraviolet limit of the free energy

$$p(\lambda) = \lim_{j_{max} \rightarrow \infty} \log Z^{j_{max}}(\lambda) \quad (2.7)$$

exists and it is a Borel summable function of λ . This paper is devoted to recover these classical results [13, 7, 14, 15] in the LVE representation. For simplicity we may continue to write C and T from now on, in which case they mean $C_{\leq j_{max}}$ and $T_{\leq j_{max}}$.

The main problem of this model compared to the toy model of [12] is that this action is not positive [13]. Observe that for $\lambda > 0$

$$e^{-\frac{\lambda}{2} \text{Tr}[(\phi^2 - 3T)^2 - 6T^2]} \leq e^{3\lambda T^2}, \quad (2.8)$$

producing Nelson's famous divergent bound [13] as $j_{max} \rightarrow \infty$:

$$|Z^{j_{max}}(\lambda)| \leq e^{\lambda O(1)j_{max}^2}. \quad (2.9)$$

Introducing the intermediate field σ , integrating out the terms that are quadratic in $\phi(x)$ and using that $\text{Tr } T\sigma = \text{Tr } C\sigma$, we get

$$Z^{j_{max}}(\lambda) = \int d\nu(\sigma) e^{\text{Tr} \left(3\lambda T^2 + 3i\sqrt{\lambda} T\sigma - \frac{1}{2} \log[1 + 2i\sqrt{\lambda} C\sigma] \right)}, \quad (2.10)$$

$$= \int d\nu(\sigma) e^{\text{Tr} \left(3\lambda T^2 + 2i\sqrt{\lambda} T\sigma - \frac{1}{2} \log_2[1 + 2i\sqrt{\lambda} C\sigma] \right)}, \quad (2.11)$$

where $d\nu(\sigma)$ is the ultralocal measure on σ with covariance $\delta(x - y)$, and the function

$$\log_2(1 - x) \equiv x + \log(1 - x) = O(x^2) \quad (2.12)$$

has to be defined in the operator sense, by the kernel:

$$[\log_2(1 + 2i\sqrt{\lambda} C\sigma)](x, y) = - \sum_{k=2}^{\infty} \frac{(-2i\sqrt{\lambda})^k}{k} \int d^2 x_1 \cdots \int d^2 x_{k-1} [C(x, x_1)\sigma(x_1)C(x_1, x_2) \cdots C(x_{k-1}, y)\sigma(y)]. \quad (2.13)$$

(Adding a trace in the left hand side of (2.13) would corresponds to multiply the right hand side by $\delta(x - y)$ and to integrate over x and y .) The perturbation theory in terms of σ is indexed by intermediate field Feynman graphs (see Figure 1) whose vertices are the loops obtained by the expansion (2.13) into traces, and whose σ -propagators, represented by wavy lines in Figure 1, correspond to the former ϕ^4 vertices of ordinary perturbation expansion, hence bear a coupling constant λ . The loop vertices are themselves cycles of the old ϕ^4 propagators, which now occur at each *corner* of the loop vertices³. We call these corner ϕ^4 propagators simply *c-propagators* for short. The perturbative order of an intermediate graph is the total number of σ -propagators. In the case of vacuum graphs, which (for simplicity) is the only one considered in this paper, it is also half the number of c-propagators. Adding sources would introduce, in addition to the loop vertices, resolvents, which can be considered as ciliated loop vertices [16]. This extension is left to the reader.

2.2 Flipping Symmetry

Remark that the initial vertex could have been decomposed in three different ways into two corners joined by a σ -propagator. It results in the existence of dualities, which in this context we could simply call *flipping symmetries*: any σ -propagator, being a δ function, can be “flipped” according to a symmetry group of order 3 by branching the four half-c-propagators into other pairs. These flips do not change the value of the associated amplitude. It means that many intermediate graphs which do not look the same have in fact the same amplitude. For instance the two first order graphs G_1 and G_2 of Figure 2 have the same value, with combinatorial weights respectively 1 and 2, leading to the possibility to express the theory at order 1 with the single graph G_1 but with combinatorial weight 3 [10].

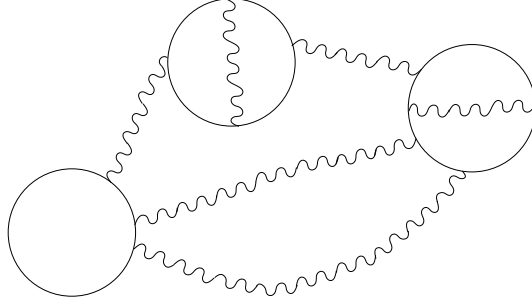


Figure 1: An Intermediate Field Graph

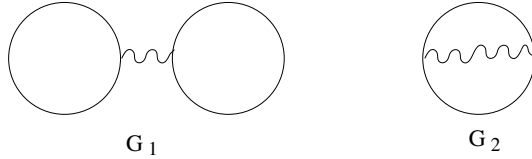


Figure 2: First Order Intermediate Field Graphs

Since the loop interaction in (2.11) is a \log_2 , the expansion into intermediate field Feynman graphs has no *loop vertex of length one*, i.e. no loop vertex with a single corner-propagator. However renormalization in the intermediate field expansion does not reduce to this observation. Wick-ordering puts to zero all tadpoles in the ordinary perturbative expansion. Accordingly, the counterterm $2i\sqrt{\lambda}T\sigma$ in (2.11) will precisely compensate any intermediate field Feynman graph in which a σ -propagator has *length one*, that is directly joins the two ends of a c-propagator. Indeed such a σ propagator of length one can be flipped into a loop vertex of length one. This is a key difficulty of the LVE in the scalar ϕ^4 theory, ultimately related to Nelson's bound (2.9). The intermediate field representation breaks the discrete dualities of the vertex, and symmetry breaking, as usual, makes renormalization more difficult ⁴.

This difficulty requires us to perform in the next subsection 2.3 an additional “slice-testing” expansion, which is the correct substitute of the ill-fated “cleaning-expansion” of [1]. For any slice j , starting with j_{max} and working up to $j = 1$, it searches for the presence of *one* c-propagator or *one* $2i\sqrt{\lambda}T_j$ counterterm of that slice. Then it performs a Wick contraction of the σ field *next* to the c-propagator C_j , and also of the σ field attached to the T_j counterterm, allowing an explicit compensation between tadpole graphs and their counterterms, ultimately bringing convergent factors $M^{-j/2}$ which will tame Nelson's bound.

However there is a difficulty. Testing for the presence of a c-propagator C_j with an interpolation parameter t_j and a Taylor expansion step does not create just a C_j . Since C_j 's occur within the $\log_2[1 + 2i\sqrt{\lambda}C\sigma]$ interaction of (2.11), derived propagators come equipped with a *resolvent*, which is the operator-valued function of the intermediate field defined by

$$R(\sigma) \equiv [1 + 2i\sqrt{\lambda}C\sigma]^{-1}. \quad (2.14)$$

The amplitudes of the combinatorial objects obtained at the end of this slice-testing expansion belong therefore to a new class, which we call *resolvent amplitudes* and which we now describe.

The resolvent amplitude A_G of an intermediate-field graph G (with slice attributions $\{j(\ell)\}$) is defined

³Remark that such intermediate field Feynman graphs are really combinatorial maps [16, 27]. It means that we can define a clockwise cyclic ordering at each loop vertex. The notion of the *next* intermediate σ field (or σ half-propagator) at any propagator is then well-defined.

⁴This difficulty occurs also in matrix models with quartic interaction, since their vertex has a duality (of order 2 instead of 3). It does not occur in vector-models [12], nor in tensor models with melonic quartic interactions [28], since their vertex has no dualities.

as

$$A_G(t, \sigma) = \prod_{v \in V(G)} [(-\lambda) \int_{[0,1]^2} d^2 x_v] \prod_{\ell \in CP(G)} [R(\sigma) C_{j(\ell)}](x_\ell, x'_\ell), \quad (2.15)$$

where $V(G)$ is the set of σ -propagators of G , $CP(G)$ is the set of c-propagators of G , and x_ℓ, x'_ℓ are the positions of the two vertices at the ends of the c-propagator ℓ .

The *renormalized* amplitude of the same graph is the same, but with subtracted tadpole resolvents:

$$A_G^R(t, \sigma) = \prod_{v \in V(G)} [(-\lambda) \int_{[0,1]^2} d^2 x_v] \prod_{\substack{\ell \in CP(G) \\ \ell \text{ tadpole}}} [(R-1)C_{j(\ell)}(\sigma)](x_\ell, x'_\ell) \prod_{\substack{\ell \in CP(G) \\ \ell \text{ not tadpole}}} [R(\sigma)C_{j(\ell)}](x_\ell, x'_\ell). \quad (2.16)$$

Hence in renormalized resolvent amplitudes, c-propagators are not ordinary C 's but either $(R-1)C$ or RC depending whether the c-propagator is a tadpole or not. They correspond therefore, when expanding the R or $R-1$ factors and performing the σ Wick contractions, to *infinite series* of ordinary ϕ^4 graphs, but with the particularity that the initial propagators C 's of the renormalized resolvent graph cannot be tadpoles. For instance at order 1 there are two resolvent graphs associated to the two intermediate graphs of Figure 2, simply replacing both c-propagators by $(R-1)C$ factors, since they are both tadpoles. Again these two graphs are equal, allowing to possibly simplify sums over resolvent graphs.

We need two more definitions. Consider any *slice-subset* $J : \{j_1, \dots, j_p\} \subset \mathcal{S} = [0, \dots, j_{max}]$. A J -resolvent graph is defined as a resolvent graph in which exactly p c-propagators bear marks $\{j_1, \dots, j_p\}$. Finally a J -resolvent graph is called *minimal* if every connected component of the graph bears at least a mark, and the total perturbative order of the graph, i.e. the total number of σ -propagators, is at most $|J|$. The set of minimal J -resolvent graphs is noted $\mathcal{G}(J)$, and we denote $\mathcal{G} = \cup_J \mathcal{G}(J)$. By convention we could say that $J = \emptyset$ is allowed, resulting in a single “empty graph” in $\mathcal{G}(\emptyset)$ with no propagator. It will correspond to the free theory, hence to the term 1 in the expansion of $Z^{j_{max}}(\lambda)$ below.

The slice-testing expansion of the next subsection will test through a Taylor remainder formula the presence of a marked (non-tadpole) propagator in any slice, hence will result in expressing the partition function of the theory as a sum over graphs in \mathcal{G} multiplying a remaining interaction. Then a factorization expansion quite similar to that of [12] can be performed. The marked propagators provide the good factors which ultimately pay for the Nelson bound and all combinatorics.

2.3 Slice-testing Expansion

We introduce inductively interpolation parameters $t_j \in [0, 1]$ for the j -th scale of the propagators.

We shall write simply t for the family $\{t_j\}, 0 \leq j \leq j_{max}$. It means that we write

$$C(t) = \sum_{j=0}^{j_{max}} t_j C_j, \quad T(t) = \sum_{j=0}^{j_{max}} t_j T_j, \quad T_j = C_j(x, x), \quad (2.17)$$

$$V(t) = \text{Tr} (3\lambda T^2(t) + 2i\sqrt{\lambda}T(t)\sigma - \frac{1}{2} \log_2[1 + 2i\sqrt{\lambda}C(t)\sigma]). \quad (2.18)$$

We perform a single first order Taylor expansion step in each t_j between 0 and 1. It results in

$$Z^{j_{max}}(\lambda) = \sum_{J \subset \mathcal{S}} \int d\nu(\sigma) \prod_{j \in J} \int_0^1 dt_j \frac{d}{dt_j} e^{V(t)}|_{t_j=0 \text{ for } j \notin J}. \quad (2.19)$$

Each $\frac{d}{dt_j}$ hits either a propagator or a tadpole, resulting in a well defined T_j or a well-defined $C_j\sigma$ brought down from the exponential. Using that $(\log_2(1+x))' = (1+x)^{-1} - 1 = -x/(1+x)$, the T_j 's comes equipped with a $T(t)$ or $\sqrt{\lambda}\sigma$, and the C_j 's comes equipped with an $R(t) - 1$, where

$$R(t) = [1 + 2i\sqrt{\lambda}C(t)\sigma]^{-1}. \quad (2.20)$$

When no confusion can result, we should shorten formulas by writing C , V or R for $C(t)$, $V(t)$ and $R(t)$.

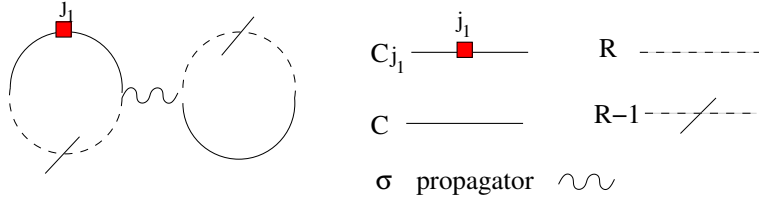


Figure 3: Minimal $G(j_1)$ resolvent graph: the single first order graph

We now explicit the result of computing (2.19) at first and second order (i.e. for $|J| = 1$ and $|J| = 2$), before describing the general case through Lemma 2.1.

Applying a first derivative term $\frac{d}{dt_{j_1}}$ in (2.19) gives

$$I_1 = \int \frac{d}{dt_{j_1}} e^V d\nu(\sigma) = \int e^V d\nu(\sigma) \text{Tr} [6\lambda T_{j_1} T + 2i\sqrt{\lambda} T_{j_1} \sigma - i\sqrt{\lambda} C_{j_1} \sigma (R-1)]. \quad (2.21)$$

To simplify the expression and explicit the cancellation with the T_j counter term we need to sandwich an integration by parts of the two σ fields hooked to C_{j_1} and T_{j_1} between the derivatives computations. Of course it does not change the value of the result, but simplifies its writing.

Contracting the two explicit σ fields in the right hand side of (2.21) gives

$$\begin{aligned} I_1 &= \int e^V d\nu(\sigma) \text{Tr} [6\lambda T_{j_1} T + 2i\sqrt{\lambda} T_{j_1} \sigma - i\sqrt{\lambda} C_{j_1} \sigma (R-1)] \\ &= \int e^V d\nu(\sigma) \left[6\lambda T_{j_1} T - 4\lambda T_{j_1} T + 2\lambda T_{j_1} \text{Tr} (R-1) C \right. \\ &\quad \left. - 2\lambda \text{Tr}_x [RC_{j_1}](x, x) [RC](x, x) + 2\lambda \text{Tr} [(R-1)C_{j_1}] T - \lambda \text{Tr} [(R-1)C_{j_1}](x, x) [(R-1)C](x, x) \right] \\ &= -3\lambda \int e^V d\nu(\sigma) \text{Tr}_x [(R-1)C_{j_1}](x, x) [(R-1)C](x, x) = \int e^V d\nu(\sigma) A_{G(j_1)}(\sigma), \end{aligned} \quad (2.22)$$

where we recall that Tr means $\int d^2x$, but we have also written more explicitly Tr_x for $\int d^2x$ when we have more than one operator at coinciding x . This expression can be labeled by two order 1 resolvent graphs corresponding to the two order 1 intermediate graphs G_1 and G_2 of Figure 2. The σ Wick contraction reconstructs a single former ϕ^4 vertex, represented as a wavy line. The two c-propagators are $(R-1)C_{j_1}$ and $(R-1)C$, because they are both of the tadpole type. C_{j_1} bears a red mark j_1 , C is a regular line without index, and we represent each $R-1$ as a dotted line.

The combinatoric weights of the two graphs are again 1 and 2. Since they have again the same value (because of the flipping symmetry or duality of the vertex) the result can be simplified as in (2.22) and expressed as a single minimal resolvent graph decorated with a single mark j_1 , hence corresponding to $|J| = 1$, which is the graph $G(j_1)$ pictured in Figure 3. Remark that the factor -3λ can be understood as $-3 \times 2(\lambda/2)$, where $\lambda/2$ is the coupling constant, (see (2.3)) 3 is the ordinary combinatoric factor for the (single) order 1 vacuum ϕ^4 graph, and the factor 2 comes from the two places in that graph where C_{j_1} can be substituted to C .

Applying a second derivative term $\frac{d}{dt_{j_2}}$ we obtain

$$\begin{aligned} I_{12} &= \int \frac{d}{dt_{j_1}} \frac{d}{dt_{j_2}} e^V d\nu(\sigma) = -3\lambda \int e^V d\nu(\sigma) \left(\text{Tr}_x \{ [(R-1)C_{j_1}](x, x) [(R-1)C_{j_2}](x, x) \right. \\ &\quad \left. - 2i\sqrt{\lambda} [(RC_{j_2} \sigma RC_{j_1}](x, x) [(R-1)C](x, x) + [(R-1)C_{j_1}](x, x) [RC_{j_2} \sigma RC](x, x) \} \right. \\ &\quad \left. + \text{Tr}_{x,y} [(R-1)C_{j_1}](x, x) [(R-1)C](x, x) [6\lambda T_{j_2} T + 2i\sqrt{\lambda} T_{j_2} \sigma - i\sqrt{\lambda} C_{j_2} \sigma (R-1)](y, y) \right) \end{aligned} \quad (2.23)$$

where $\text{Tr}_{x,y}$ means $\int d^2x d^2y$ (beware to use the correct order for composition of operators!).

We can keep the term

$$J_{12} = -3\lambda \int e^V d\nu(\sigma) [(R-1)C_{j_1}](x, x) [(R-1)C_{j_2}](x, x) = \int e^V d\nu(\sigma) A_{G_1(j_1, j_2)}(\sigma) \quad (2.24)$$

without further change. It is the only minimal (j_1, j_2) -decorated resolvent graph at order 1, pictured as $G_1(j_1, j_2)$ in Figure 4.

The contraction of the explicit σ field in the other terms in (2.23) again allows for the cancellation of tadpoles, some obvious and others requiring attention. To organize the computation we remark that in the last term, the contraction of the sigma field inside the same trace or to the exponential reproduces the previous computation. Hence we obtain a relatively simple term which correspond to a disconnected graph, namely the graph $G_2(j_1, j_2)$ in Figure 4 whose value is

$$\begin{aligned} K_{12} &= 9\lambda^2 \int e^V d\nu(\sigma) \text{Tr}_x [(R-1)C_{j_1}](x, x) [(R-1)C](x, x) \text{Tr}_y [(R-1)C_{j_2}](y, y) [(R-1)C](y, y) \\ &= \int e^V d\nu(\sigma) A_{G(j_1)}(\sigma) A_{G(j_2)}(\sigma). \end{aligned} \quad (2.25)$$

We remark that the integrand for a disconnected such resolvent graph factorizes over its connected components, as for usual Feynman graphs.

Then writing $I_{12} = J_{12} + K_{12} + L_{12}$ we obtain a lengthy sum of terms for L_{12} , which correspond to second order connected graphs (omitting the trivial Tr_x and Tr_y integrations for simplicity):

$$\begin{aligned} L_{12} &= 6\lambda^2 \int e^V d\nu(\sigma) \{ -2T([RC_{j_2}RC_{j_1}](x, x)[(R-1)C](x, x) + [(R-1)C_{j_1}](x, x)[RC_{j_2}RC](x, x)) \\ &+ [(R-1)C](x, x)([RC](x, y)[RC_{j_2}](y, x)[(R-1)C_{j_1}](y, y) + [RC_{j_1}](x, y)[RC_{j_2}](y, x)[(R-1)C](y, y)) \\ &+ 2[(R-1)C_{j_1}](x, x)([RC](x, y)[RC](y, y)[RC_{j_2}](y, x) + [RC](x, y)[RC_{j_2}](y, y)[RC](y, x)) \\ &+ 2[(R-1)C](x, x)([RC_{j_1}](x, y)[RC](y, y)[RC_{j_2}](y, x) + [RC_{j_1}](x, y)[RC_{j_2}](y, y)[RC](y, x)) \\ &+ 2[RC](x, y)[RC_{j_1}](y, x)[RC](x, y)[RC_{j_2}](y, x) + 2[RC](x, y)[RC](y, x)[RC_{j_1}](x, y)[RC_{j_2}](y, x) \\ &- 2T_{j_2}([RC_{j_1}RC](x, x)[(R-1)C](x, x) + [RCRC](x, x)[(R-1)C_{j_1}](x, x)) \\ &+ [(R-1)C_{j_2}](x, x)([RC](x, y)[RC_{j_1}](y, x)[(R-1)C](y, y) + [RC](x, y)[RC](y, x)[(R-1)C_{j_1}](y, y)) \}. \end{aligned} \quad (2.26)$$

We can now cancel the first and sixth lines of this equation (which contains tadpoles T and T_{j_2}) respectively with the first term of the resolvent expansion of the terms $CR(x, x)$ and $C_{j_2}R(x, x)$ in lines 3 and 4. It renormalized the corresponding tadpoles, exactly transforming the corresponding R factors into $(R-1)$ factors. Hence we get a tadpole-free expression:

$$\begin{aligned} L_{12} &= 6\lambda^2 \int e^V d\nu(\sigma) \{ \\ &+ [(R-1)C](x, x)([RC](x, y)[RC_{j_2}](y, x)[(R-1)C_{j_1}](y, y) + [RC_{j_1}](x, y)[RC_{j_2}](y, x)[(R-1)C](y, y)) \\ &+ 2[(R-1)C_{j_1}](x, x)([RC](x, y)[(R-1)C](y, y)[RC_{j_2}](y, x) + [RC](x, y)[(R-1)C_{j_2}](y, y)[RC](y, x)) \\ &+ 2[(R-1)C](x, x)([RC_{j_1}](x, y)[(R-1)C](y, y)[RC_{j_2}](y, x) + [RC_{j_1}](x, y)[(R-1)C_{j_2}](y, y)[RC](y, x)) \\ &+ 2[RC](x, y)[RC_{j_1}](y, x)[RC](x, y)[RC_{j_2}](y, x) + 2[RC](x, y)[RC](y, x)[RC_{j_1}](x, y)[RC_{j_2}](y, x) \\ &+ [(R-1)C_{j_2}](x, x)([RC](x, y)[RC_{j_1}](y, x)[(R-1)C](y, y) + [RC](x, y)[RC](y, x)[(R-1)C_{j_1}](y, y)) \} \end{aligned} \quad (2.27)$$

Using the flipping symmetry, we can simplify it slightly and rewrite it as

$$\begin{aligned} L_{12} &= 6\lambda^2 \int e^V d\nu(\sigma) \{ 3[(R-1)C](x, x)[RC_{j_1}](x, y)[RC_{j_2}](y, x)[(R-1)C](y, y) \\ &+ 3[(R-1)C](x, x)[RC](x, y)[RC_{j_1}](y, x)[(R-1)C_{j_2}](y, y) \\ &+ 3[(R-1)C](x, x)[RC](x, y)[RC_{j_2}](y, x)[(R-1)C_{j_1}](y, y) \\ &+ 3[(R-1)C_{j_1}](x, x)[RC](x, y)[RC](y, x)[(R-1)C_{j_2}](y, y) \\ &+ 2[RC](x, y)[RC_{j_1}](y, x)[RC](x, y)[RC_{j_2}](y, x) + 2[RC](x, y)[RC](y, x)[RC_{j_1}](x, y)[RC_{j_2}](y, x) \}. \end{aligned}$$

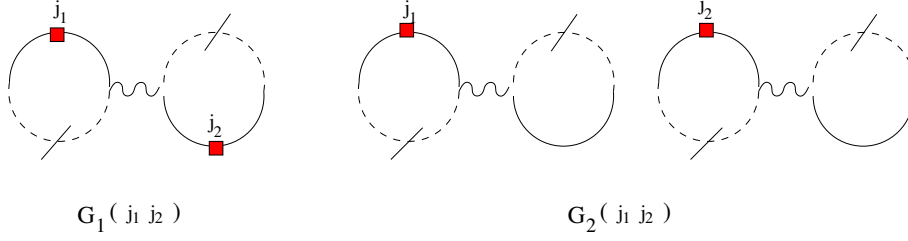


Figure 4: Minimal $\{j_1, j_2\}$ resolvent graphs: the first order graph and the second order disconnected graph

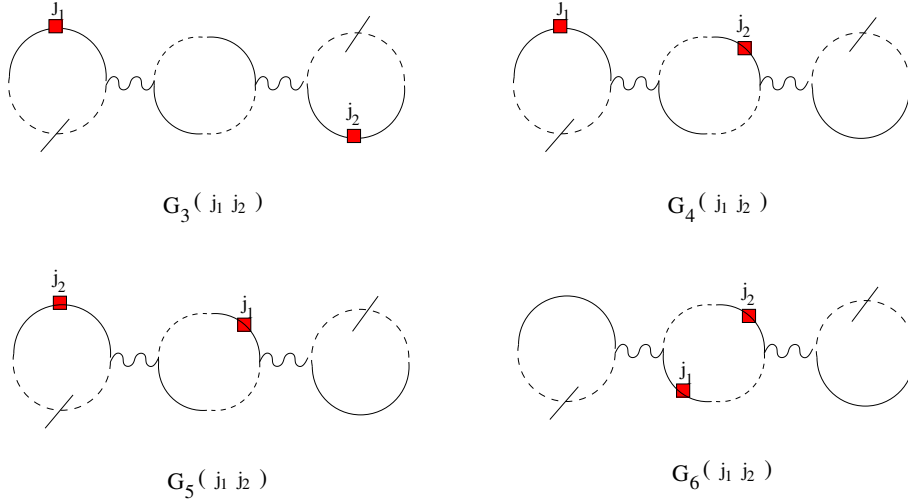


Figure 5: Minimal (j_1, j_2) -decorated resolvent graphs: second order connected graphs

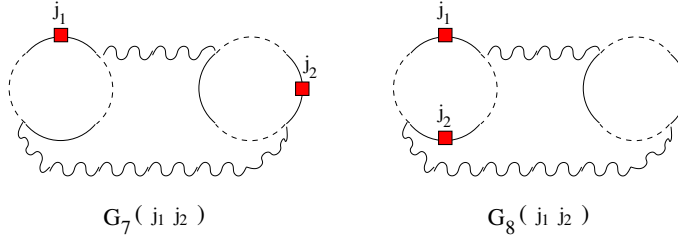


Figure 6: Minimal (j_1, j_2) -decorated resolvent graphs: second order connected graphs (end)

The result is indexed by the graphs $G_i(j_1, j_2)$, $i = 3, \dots, 6$ in Figures 4 and 5 which have weight $18\lambda^2$ and of graphs $G_i(j_1, j_2)$, $i = 7, 8$ in Figure 6, with weight $12\lambda^2$. Remark that although there are a priori more intermediate field graphs at order 2 than shown in these figures, because of duality, using the flipping symmetry of the σ propagator we have reexpressed the result in terms of the fewer graphs of Figures 4, 5 and 6.

Collecting these results we have obtained the sum over all vacuum Feynman graphs of order 1 and 2 with ordinary propagators CR and tadpole propagators $C(R-1)$, in which one propagator is marked j_1 and the other marked j_2 , in all possible ways, with their natural symmetry factors, with just one further restriction, namely that every connected component of G must bear at least one mark (see Figures 4-6).

The key step is the cancellation of the σ fields associated to T or T_j factors with the tadpoles created by contractions of the σ fields associated to C_j or C propagators to their own neighbor in the same loop.

This cannot be an accident. The intermediate field representation represents exactly the same graphs as the initial theory, in which no tadpole can exist, and since the slice expansion tests blindly the presence at first order of a given scale j anywhere in the theory, either within graphs or counterterms, without *any constraint*, this cancellation *must happen*. The next lemma states this for arbitrarily large $|J|$.

Lemma 2.1 *The general term of the testing expansion is*

$$Z^{jmax}(\lambda) = \sum_{J \subset S} \sum_{G \in \mathcal{G}(J)} c_G \int d\nu(\sigma) \prod_{j \in J(G)} \int_0^1 dt_j [e^{V(t)} A_G^R(t, \sigma)]_{t_j=0 \text{ for } j \notin J(G)} \quad (2.28)$$

where the sum over $\mathcal{G}(J)$ runs over a set of minimal (vacuum) resolvent graphs (connected or not). The renormalized amplitudes A_G^R are defined by (2.16), where the index $j(\ell)$ specifies the markings, that is restricts the c -propagator ℓ to be C_j if that propagator bears the mark j . All marked propagators which belong to a short cycle (i.e. potential tadpole) are renormalized, hence accompanied by an $R - 1$ resolvent factor, and not an R factor. The c -propagators which do not bear any mark are equal to $C(t)$.

Proof

The key step is to generalize the cancellation from (2.26) to (2.27). By induction on $n = |J|$ we suppose Lemma 2.1 holds up to n slices $J = \{j_1, \dots, j_n\}$, and we add a new $d/dt_{j_{n+1}}$ and contract the explicit σ fields it created. If this process creates a new disconnected graph, it is the first step computation (2.22) and we get a new marked and renormalized connected component $G(j_{n+1})$. The other terms decompose into:

A) terms with exactly one explicit $T_{j_{n+1}}$ counterterm, in which the σ associated to $T_{j_{n+1}}$ has contracted into a resolvent of a graph created by the previous steps,

B) terms with exactly one explicit $\text{Tr } RC_{j_{n+1}}$ ("potential tadpole" of scale $j + 1$).

- other terms with no $T_{j_{n+1}}$ counterterm and no potential tadpole of any scale j_1, \dots, j_{n+1} .

Developing the potential tadpoles as $\text{Tr } RC_{j_{n+1}} = T_{j_{n+1}} + \text{Tr } (R - 1)C_{j_{n+1}}$ transforms them into singular $T_{j_{n+1}}$ parts plus renormalized tadpoles $\text{Tr } (R - 1)C_{j_{n+1}}$. The Lemma states that the A terms exactly cancel the singular B terms. This can be checked inductively by a painful analysis of cases where σ contract left or right. But we prefer a more abstract argument. Remark that we could apply our expansion to *any* scale decomposition of the propagator into any finite subset of scales (not necessarily *contiguous*) and the algebra would always be the same. If we then put all t_{j_k} , $k = 1, \dots, n+1$, parameters to zero in every term after our j_{n+1} expansion step and keep a last remaining (infrared) $C = C_0$ scale, what we obtain must be the ϕ_2^4 : theory with exactly a single propagator or tadpole of each scale j_1, \dots, j_{n+1} and an arbitrary number of C_0 propagators and tadpoles. Keeping the scales j_1, \dots, j_n all fixed and letting $j_{n+1} \rightarrow \infty$ proves that the A and singular B coefficients exactly cancel, otherwise the corresponding ϕ_2^4 : perturbative theory restricted to that sequence of scales and to first order in $C_{j_1}, \dots, C_{j_{n+1}}$ would not be perturbatively finite (as it must be). \square

With a little more work we could also prove that the terms with exactly one explicit T counterterm and the singular part of the terms with exactly one explicit $\text{Tr } RC$ factor also exactly cancel each other, as seen in the computations above with one or two scales. Since we do not use this refinement in what follows, we leave such details to the reader. Using the flipping symmetry we can also simplify the result to express it in terms of a minimal number of intermediate field graphs. This cannot play an important constructive role since the orbits at order n have at most 3^n terms, hence do not change any of the constructive bounds. The combinatoric factors c_G for the sum (2.28) are the natural ones so that putting resolvent factors to 1 one recovers the right factors of ϕ^4 perturbation theory up to order $|J|$, but the precise rules to compute them can be a bit obscured by the use of the flipping symmetry and are not essential for constructive purpose (they can be bounded easily). Hence we leave them also to the reader.

2.4 Factorization of Interaction

Consider a fixed minimal resolvent graph $G \in \mathcal{G}$ corresponding to a non-empty set⁵ of marks J and to parameters t_j , $j \in J$. We need now also in (2.28) to factorize the action $e^{V(t)}$ into pieces attributed to each slice⁶.

⁵Recall that an empty set of marks correspond to the term 1 in the expansion of $Z^{jmax}(\lambda)$.

⁶If we were to interchange this factorization and the testing expansion, the cancellation of tadpoles with counterterms would not be exact.

In order to perform this factorization we shall attribute to each loop vertex the index of its highest c-propagator. To write explicitly the result, hence the piece of the loop vertex attributed to slice j , we introduce an additional auxiliary parameter u_j , $j \in J$ which multiplies t_j (we could also say that we substitute $t_j(u_j) = u_j t_j$ to t_j) and we rewrite the interaction with cutoff j as

$$V_{\leq j}(t, \sigma) \equiv \text{Tr} \left[3\lambda T_{\leq j}^2(t) + 2i\sqrt{\lambda} T_{\leq j}(t)\sigma - \frac{1}{2} \log_2[1 + 2i\sqrt{\lambda} C_{\leq j}(t)\sigma] \right]_{u_j=1}. \quad (2.29)$$

The specific part of the interaction which should be attributed to the scale j is the sum over all loop vertices with at least one c-propagator at scale j and all others at scales $\leq j$. Hence it is

$$\begin{aligned} V_j &\equiv V_{\leq j} - V_{\leq j-1} = V_{\leq j}|_{u_j=1} - V_{\leq j}|_{u_j=0} \\ &= \int_0^1 du_j \frac{d}{du_j} \text{Tr} \left[3\lambda T_{\leq j}^2(t_j) + 2i\sqrt{\lambda} T_{\leq j}(t_j)\sigma - \frac{1}{2} \log_2[1 + 2i\sqrt{\lambda} C_{\leq j}(t_j)\sigma] \right] \\ &= \int_0^1 t_j du_j \text{Tr} \left[6\lambda T_j T_{\leq j}(t) + 2i\sqrt{\lambda} T_j \sigma - i\sqrt{\lambda} [R_{\leq j}(t) - 1] C_j \sigma \right] \end{aligned} \quad (2.30)$$

where we recall that the resolvent is

$$R_{\leq j}(t) = (1 + 2i\sqrt{\lambda} C_{\leq j}(t)\sigma)^{-1}. \quad (2.31)$$

Remark that since $t_j = 0$ for $j \notin J$, the total interaction can be written as

$$V_{\leq j_{\max}} = \sum_{j=0}^{j_{\max}} V_j = \sum_{j \in J} V_j, \quad (2.32)$$

since all terms V_j with $j \notin J$ are put to zero by the condition $t_j = 0$.

To each graph $G \in \mathcal{G}(J)$ and $J \subset \mathcal{S}$ is associated a partition into the (non-empty) connected components G_1, \dots, G_n of G and an associated partition J_1, \dots, J_n where $J_a \subset \mathcal{S}$ is the (non-empty) set of marks present in G_a . We have $J_a \cap J_b = \emptyset$ for any $a \neq b$. Since combinatoric weights of Feynman graph factor over connected components, $c_G = \prod_a c_{G_a}$ and we can rewrite the result of the slice-testing expansion as

$$Z^{j_{\max}}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu(\sigma) \sum_{J_1, \dots, J_n, J_a \cap J_b = \emptyset} \left[\prod_{a=1}^n \prod_{j \in J_a} \int_0^1 dt_j \right] \prod_{a=1}^n \mathcal{I}(J_a, t, \sigma) \quad (2.33)$$

where

$$\mathcal{I}(J_a, t, \sigma) = \sum_{G_a \in \mathcal{CG}(J_a)} c_{G_a} \left(\prod_{j \in J_a} [e^{V_j(t, \sigma)}] A_{G_a}^R(t, \sigma) \right), \quad (2.34)$$

$\mathcal{CG}(J)$ being the set of *connected* graphs in $\mathcal{G}(J)$. The $1/n!$ in (2.33) comes from summing over *ordered* sequences J_1, \dots, J_n in (2.33). The sum over n is in fact not infinite since the hardcore constraint forces all terms to be zero for $n > 1 + j_{\max}$. The term $n = 0$ corresponds to the factor 1 in the sum (the normalization of the free theory).

We would like to factorize the sums over the blocks J_a . The functions \mathcal{I} are still coupled through the σ functional integral, the hardcore constraint $J_a \cap J_b = \emptyset$ and the t parameters. Indeed the parameters t in $\mathcal{I}(J, t, \sigma)$ are not restricted to those of the block J ⁷. The first two issues are rather standard and in the next section we shall treat them with the strategy of [12]. We now address the last issue through an *auxiliary expansion*.

Consider an integral such as

$$F = \int dt_1 \cdots dt_n \prod_{a=1}^n f_a(\vec{t}_1, \dots, \vec{t}_n) \quad (2.35)$$

⁷We are very grateful to an anonymous referee for pointing this subtlety to us, as we missed it in an earlier version of the paper.

We can introduce for each pair $1 \leq a < b \leq n$ a pair interpolation parameters $x_{a,b}$ to simultaneously multiply the t_b dependence of f_a and the t_a dependence of f_b , so we write

$$F = \int d\vec{t}_1 \cdots d\vec{t}_n \prod_{a=1}^n f_a(x_{1,a}\vec{t}_1, \dots, x_{a-1,a}\vec{t}_{a-1}, \vec{t}_a, x_{a,a+1}\vec{t}_{a+1}, \dots, x_{a,n}\vec{t}_n) |_{x_{a,b}=1 \forall a,b} \quad (2.36)$$

The forest formula of [5, 6] applied to F gives

$$F = \sum_{\mathcal{F}} \int d\vec{t}_1 \cdots d\vec{t}_n \int dw_{\mathcal{F}} \partial^{\mathcal{F}} \prod_{a=1}^n f_a(x_{1,a}\vec{t}_1, \dots, x_{a-1,a}\vec{t}_{a-1}, \vec{t}_a, x_{a,a+1}\vec{t}_{a+1}, \dots, x_{a,n}\vec{t}_n) |_{x_{a,b}=x_{a,b}^{\mathcal{F}}(\{w\}) \forall a,b} \quad (2.37)$$

where the sum is over all *forests* \mathcal{F} on $(1, \dots, n)$, $\int dw_{\mathcal{F}}$ means $\prod_{\ell=(a,b) \in \mathcal{F}} \int_0^1 dw_{a,b}$, $\partial^{\mathcal{F}}$ means $\prod_{\ell=(a,b) \in \mathcal{F}} \frac{\partial}{\partial x_{a,b}}$ and $x_{a,b}^{\mathcal{F}}\{w\}$ is as usual the infimum of the $w_{a',b'}$'s on the path of links (a', b') in \mathcal{F} joining a to b (0 if no such path exists) [5, 6].

Let us apply this formula to (2.33) and to $f_a = \mathcal{I}(J_a, t, \sigma)$, with \vec{t}_a being the list of $|J_a|$ parameters t_j for $j \in J_a$. We have to compute the action of $\partial^{\mathcal{F}}$ on f . Any derivative $\frac{\partial}{\partial x_{a,b}}$ in $\partial^{\mathcal{F}}$ can act either on an $e^{V_j(t, \sigma)}$ factor or on the propagators resolvents of $A_{G_a}^R(t, \sigma)$. In the first case a derivative creates either a new loop vertex $\frac{\partial}{\partial t_j} V_{j'}$ = of any scale $j \in J_a$ for any $j' \in J_b$ or the converse (exchange a with b in the sentence). each derived (crossed) C propagator comes equipped with a σR factor. They can also act on the propagators C_j for any $j \in J_b$ in the propagators or resolvents of $A_{G_a}^R(t, \sigma)$, or the converse (exchange a with b in the sentence). Hence they always add a new propagator C_j or C_i or a tadpole $3\lambda T_j T_i$ by hitting $3\lambda T^2$. These new propagators or tadpoles are crossed to distinguish them from the previous marked ones. Note, although this is not essential, that the auxiliary expansion cannot act on the $2i\sqrt{\lambda}T_j$ term in (2.30), since it contains a single frequency j , hence cannot be interpolated by any x_{ab} parameter.

The important point is that for this auxiliary expansion we make absolutely no attempt to cancel any tadpoles with counterterms, Such cancellations are no longer necessary because we already gathered extremely strong convergence factors for ultraviolet convergence through the marked renormalized propagators of the previous t expansion, that can easily absorb a rather large number of logarithmic divergencies.

The result is a sum over new larger blocks J of indices, which are finite unions of p previous blocks J_a , those corresponding to the connected components of \mathcal{F} , hence to trees \mathcal{T} of \mathcal{F} with $p-1$ edges. The associated modified amplitudes $\mathcal{IM}(J, t, \sigma)$ contain

- an integral $\prod_{j \in J} \int_0^1 dt_j$ which we note $\int dt_J$ for short,
- a sum over all partitions Π of J into finite disjoint non empty blocks $J = J_{a_1} \cup \dots \cup J_{a_p}$,
- a sum over all trees \mathcal{T} on $\{1, p\}$,
- an integral $\int dw_{\mathcal{T}}$ over w parameters,
- the exponential of the interaction factor $\prod_{j \in J} [e^{V_j(t, \sigma)}]$, where the t 's are now multiplied by their appropriate $x_{a,b}^{\mathcal{T}}(\{w\})$ factors,
- a sum over G in the set $\mathcal{CGM}(J, \Pi, \mathcal{T})$ indexing all the ways the derivatives in $\partial^{\mathcal{T}}$ acted, of a corresponding modified amplitude $A_G^{R,M}(t, \sigma)$. The modified graph G can be pictured as containing marked renormalized propagators C_j , crossed propagators and/or counterterms plus ordinary propagators and R or $R-1$ resolvents. Its amplitude is obtained by collecting all the terms brought down from the exponential both by the slice testing expansion which created the blocks J_{a_1}, \dots, J_{a_p} and by the auxiliary expansion which joined J_{a_1}, \dots, J_{a_p} . The corresponding modified graphs have the following characteristics
 - they are no longer necessarily connected,
 - they contain exactly one renormalized marked propagator C_j for every $j \in J$ (hence equipped with an $R-1$ factor if in a short loop of length 1)

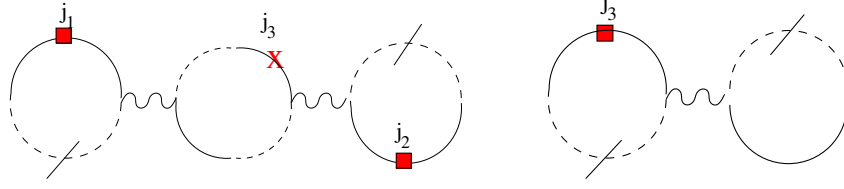


Figure 7: A modified graph connecting the block $J_a = \{j_1, j_2\}$ to the block $J_b = \{j_3\}$

- they contain at most $|J| - 1$ crossed propagators or T_j counterterms, each with a scale index $j \in J$. A crossed propagator is preceded by an R or $R - 1$ resolvent.
- their order of perturbation (number of explicit λ factors not inside resolvents) is bounded by $4|J|$

This is illustrated in Figure 7, in which two blocks $J_a = \{j_1, j_2\}$ and $J_b = \{j_3\}$ are linked by a d/dt_{j_3} , acting in fact on a graph $G_3(j_1, j_2)$ of Figure 5. It decorated that graph with a crossed propagator of scale j_3 . The resulting graph for the new block $\{j_1, j_2, j_3\}$ is disconnected and still bears three marked renormalized propagators, one for each scale of $\{j_1, j_2, j_3\}$. This is enough to ensure excellent ultraviolet convergence of the new block, even if we do not care (during auxiliary expansion steps) to compensate short cycles of length 1 (potential tadpoles) with their associated crossed counterterms.

We can therefore write a modified formula similar to (2.34), with hopefully clear notations

$$Z^{j_{max}}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu(\sigma) \sum_{J_1, \dots, J_n, J_a \cap J_b = \emptyset} \prod_{a=1}^n \mathcal{IM}(J_a, \sigma) \quad (2.38)$$

where

$$\mathcal{IM}(J, \sigma) = \sum_{\Pi} \sum_{\mathcal{T}} \int dt_J \int dw_{\mathcal{T}} \sum_{G \in \mathcal{CGM}(J, \Pi, \mathcal{T})} c_G \left(\prod_{j \in J} [e^{V_j(t, w, \sigma)}] A_G^{R, M}(t, w, \sigma) \right). \quad (2.39)$$

The w dependence means that all t factors are multiplied by their appropriate unessential $x_{a,b}^T(\{w\})$ factors.

In (2.39) the t integrals now have obviously factorized over the new blocks. That was the sole purpose of the auxiliary expansion. We can now encode the hardcore constraints $J_a \cap J_b = \emptyset$ through Grassmann numbers as

$$Z^{j_{max}}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu(\sigma) d\mu_{\mathbb{I}_S}(\bar{\chi}, \chi) \prod_{a=1}^n \left(\sum_{J_a \subset S} \sum_{\Pi} \sum_{\mathcal{T}} \int dt_{J_a} \int dw_{\mathcal{T}} \sum_{G \in \mathcal{CGM}(J, \Pi, \mathcal{T})} c_G \left(\prod_{j \in J_a} [-\bar{\chi}_j e^{V_j(t, w, \sigma)} \chi_j] A_G^{R, M}(t, w, \sigma) \right) \right) \quad (2.40)$$

where $d\mu_{\mathbb{I}_S}(\bar{\chi}, \chi) = \prod_{j=0}^{j_{max}} d\bar{\chi}_j d\chi_j e^{-\bar{\chi}_j \chi_j}$ is the standard normalized Grassmann Gaussian measure with covariance \mathbb{I}_S , the $1 + j_{max}$ by $1 + j_{max}$ identity matrix. Indeed Grassmann Gaussian variables automatically implement the hardcore constraints, and saturate the Grassmann pairs for $j \notin J = \cup_a J_a$.

We can remark that (2.40) is the developed expansion for a new type of vertex W which is a sum over slice-subsets J :

$$Z^{j_{max}}(\lambda) = \int d\nu(\sigma) d\mu_{\mathbb{I}_S}(\bar{\chi}, \chi) e^W, \quad (2.41)$$

$$W = \sum_{J \subset S} \sum_{\Pi} \sum_{\mathcal{T}} \int dt_J \int dw_{\mathcal{T}} \sum_{G \in \mathcal{CGM}(J, \Pi, \mathcal{T})} c_G \left(\prod_{j \in J} [-\bar{\chi}_j e^{V_j(t, w, \sigma)} \chi_j] A_G^{R, M}(t, w, \sigma) \right) \quad (2.42)$$

To distinguish W , which contains potentially infinitely many vertices of type V , we call it an *exp-vertex*.

From now on and until subsection 4.5, in order not to distract the reader we drop in the writing of W all the auxiliary expansion $\sum_{\Pi} \sum_{\mathcal{T}}$ and w factors and the M index which recall that graphs G can contain

crossed propagators and/or crossed T counterterms. Indeed these auxiliary factors play absolutely no role, until we return to them, at the end of subsection 4.5, to see that they only trivially modify the unessential constants of the final bounds.

This ends the preparation phase and we can now proceed to the MLVE expansion proper.

3 The Multiscale Loop Vertex Expansion

We perform now the two-level jungle expansion defined in [12], starting from (2.40) which is the developed form of (2.41). For completeness we summarize the main steps, referring to [12] for details.

The first step introduces Bosonic replicas for all the exp-vertices in (2.40). Noting $\mathcal{W} = \{1, \dots, n\}$ the set of labels for these exp-vertices, we have

$$Z^{jmax}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu_{\mathcal{W}}(\sigma, \bar{\chi}, \chi) \prod_{a=1}^n W_a(\sigma_a, \bar{\chi}, \chi), \quad (3.1)$$

so that each vertex W_a has now its own Bosonic field σ^a . The replicated measure is completely degenerate between replicas:

$$d\nu_{\mathcal{W}} = d\nu_{\mathbf{1}_{\mathcal{W}}}(\{\sigma_a\}) d\mu_{\mathbb{I}_S}(\bar{\chi}_j, \chi_j) \quad (3.2)$$

$$W_a(\sigma_a, \bar{\chi}, \chi) = \sum_{J \subset S} \int dt_J \sum_{G \in \mathcal{CG}(J)} c_G \left(\prod_{j \in J} [-\bar{\chi}_j e^{V_j(t, \sigma_a)} \chi_j] A_G^R(t, \sigma_a) \right). \quad (3.3)$$

where $\mathbf{1}_{\mathcal{W}}$ is the n by n matrix with coefficients 1 everywhere.

The obstacle to factorize the functional integral Z over vertices and to compute $\log Z$ lies in the Bosonic degenerate blocks $\mathbf{1}_{\mathcal{W}}$ and in the Fermionic fields which couple the vertices W_a . In order to remove these two obstacles we need to apply *two* successive forest formulas [5, 6], one Bosonic, the other Fermionic.

To analyze the block $\mathbf{1}_{\mathcal{W}}$ in the measure $d\nu$ we introduce coupling parameters $x_{ab} = x_{ba}, x_{aa} = 1$ between the Bosonic vertex replicas and obtain a new sum over forests. Representing Gaussian integrals as derivative operators as in [12] we have

$$Z^{jmax}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[e^{\frac{1}{2} \sum_{a,b=1}^n x_{ab} \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} + \sum_{j=0}^{jmax} \frac{\partial}{\partial \bar{\chi}_j} \frac{\partial}{\partial \chi_j}} \prod_{a=1}^n W_a(\sigma_a, \bar{\chi}, \chi) \right]_{\substack{\sigma, \chi, \bar{\chi}=0 \\ x_{ab}=1}}. \quad (3.4)$$

The next step applies the standard Taylor forest formula of [5, 6] to the x parameters. We denote by \mathcal{F}_B a Bosonic forest with n vertices labelled $\{1, \dots, n\}$. It means an acyclic set of edges over \mathcal{W} . For ℓ_B a generic edge of the forest we denote by $a(\ell_B), b(\ell_B)$ the end vertices of ℓ_B . The result of the Taylor forest formula is:

$$\begin{aligned} Z^{jmax}(\lambda) = & \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}_B} \int_0^1 \left(\prod_{\ell_B \in \mathcal{F}_B} dw_{\ell_B} \right) \left[e^{\frac{1}{2} \sum_{a,b=1}^n X_{ab}(w_{\ell_B}) \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} + \sum_{j=0}^{jmax} \frac{\partial}{\partial \bar{\chi}_j} \frac{\partial}{\partial \chi_j}} \right. \\ & \times \left. \prod_{\ell_B \in \mathcal{F}_B} \left(\frac{\partial}{\partial \sigma_{a(\ell_B)}} \frac{\partial}{\partial \sigma_{b(\ell_B)}} \right) \prod_{a=1}^n W_a(\sigma_a, \bar{\chi}, \chi) \right]_{\sigma, \chi, \bar{\chi}=0}, \end{aligned}$$

where $X_{ab}(w_{\ell_B})$ is the infimum over the parameters w_{ℓ_B} in the unique path in the forest \mathcal{F}_B connecting a to b . This infimum is set to 1 if $a = b$ and to zero if a and b are not connected by the forest [5, 6].

The forest \mathcal{F}_B partitions the set of vertices into blocks \mathcal{B} corresponding to its connected components. In each such block the edges of \mathcal{F}_B form a spanning tree. Remark that such blocks can be reduced to bare vertices. Any vertex a belongs to a unique Bosonic block \mathcal{B} . Contracting every Bosonic block to an “effective vertex” we obtain a graph which we denote $\{n\}/\mathcal{F}_B$.

The next step introduces replica Fermionic fields $\chi_j^{\mathcal{B}}$ for these blocks of \mathcal{F}_B (i.e. for the effective vertices of $\{n\}/\mathcal{F}_B$) and replica coupling parameters $y_{\mathcal{B}\mathcal{B}'} = y_{\mathcal{B}'\mathcal{B}}$. The last step applies (once again) the forest formula, this time for the y 's, leading to a set of Fermionic edges \mathcal{L}_F forming a forest in $\{n\}/\mathcal{F}_B$ (hence connecting Bosonic blocks). Denoting L_F a generic Fermionic edge connecting blocks and $\mathcal{B}(L_F), \mathcal{B}'(L_F)$

the end blocks of the Fermionic edge L_F we follow exactly the same steps than in [12] and obtain a two-level-jungle formula [6]. It writes

$$Z^{j_{max}}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{J}} \int dw_{\mathcal{J}} \int d\nu_{\mathcal{J}} \partial_{\mathcal{J}} \left[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} W_a(\sigma_a, \chi^{\mathcal{B}}, \bar{\chi}^{\mathcal{B}}) \right], \quad (3.5)$$

where

- the sum over \mathcal{J} runs over all two-level jungles, hence over all ordered pairs $\mathcal{J} = (\mathcal{F}_B, \mathcal{F}_F)$ of two (each possibly empty) disjoint forests on \mathcal{W} , such that \mathcal{F}_B is a forest, \mathcal{F}_F is a forest and $\bar{\mathcal{J}} = \mathcal{F}_B \cup \mathcal{F}_F$ is still a forest on \mathcal{W} . The forests \mathcal{F}_B and \mathcal{F}_F are the Bosonic and Fermionic components of \mathcal{J} .
- $\int dw_{\mathcal{J}}$ means integration from 0 to 1 over parameters w_{ℓ} , one for each edge $\ell \in \bar{\mathcal{J}}$, namely $\int dw_{\mathcal{J}} = \prod_{\ell \in \bar{\mathcal{J}}} \int_0^1 dw_{\ell}$. There is no integration for the empty forest since by convention an empty product is 1. A generic integration point $w_{\mathcal{J}}$ is therefore made of $|\bar{\mathcal{J}}|$ parameters $w_{\ell} \in [0, 1]$, one for each $\ell \in \bar{\mathcal{J}}$.

$$\partial_{\mathcal{J}} = \prod_{\substack{\ell_B \in \mathcal{F}_B \\ \ell_B = (a, b)}} \left(\frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} \right) \prod_{\substack{\ell_F \in \mathcal{F}_F \\ \ell_F = (d, e)}} \sum_{j_{\ell_F}=0}^{j_{max}} \left(\frac{\partial}{\partial \bar{\chi}_{j_{\ell_F}}^{\mathcal{B}(d)}}} \frac{\partial}{\partial \chi_{j_{\ell_F}}^{\mathcal{B}(e)}}} + \frac{\partial}{\partial \bar{\chi}_{j_{\ell_F}}^{\mathcal{B}(e)}}} \frac{\partial}{\partial \chi_{j_{\ell_F}}^{\mathcal{B}(d)}}} \right), \quad (3.6)$$

where $\mathcal{B}(d)$ denotes the Bosonic block to which the vertex d belongs.

- The measure $d\nu_{\mathcal{J}}$ has covariance $X(w_{\ell_B})$ on Bosonic variables and $Y(w_{\ell_F}) \otimes \mathbb{I}_{\mathcal{S}}$ on Fermionic variables, hence

$$\int d\nu_{\mathcal{J}} F = \left[e^{\frac{1}{2} \sum_{a,b=1}^n X_{ab}(w_{\ell_B}) \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} + \sum_{\mathcal{B}, \mathcal{B}'} Y_{\mathcal{B}\mathcal{B}'}(w_{\ell_F}) \sum_{j_{\mathcal{B}}, j_{\mathcal{B}'}} \frac{\partial}{\partial \bar{\chi}_{j_{\mathcal{B}}}^{\mathcal{B}}} \frac{\partial}{\partial \chi_{j_{\mathcal{B}'}}^{\mathcal{B}'}}} F \right]_{\sigma=\bar{\chi}=\chi=0}. \quad (3.7)$$

- $X_{ab}(w_{\ell_B})$ is the infimum of the w_{ℓ_B} parameters for all the Bosonic edges ℓ_B in the unique path $P_{a \rightarrow b}^{\mathcal{F}_B}$ from a to b in \mathcal{F}_B . This infimum is set to zero if such a path does not exist and to 1 if $a = b$.
- $Y_{\mathcal{B}\mathcal{B}'}(w_{\ell_F})$ is the infimum of the w_{ℓ_F} parameters for all the Fermionic edges ℓ_F in any of the paths $P_{a \rightarrow b}^{\mathcal{F}_B \cup \mathcal{F}_F}$ from some vertex $a \in \mathcal{B}$ to some vertex $b \in \mathcal{B}'$. This infimum is set to 0 if there are no such paths, and to 1 if such paths exist but do not contain any Fermionic edges.

Remember that a main property of the forest formula is that the symmetric n by n matrix $X_{ab}(w_{\ell_B})$ is positive for any value of $w_{\mathcal{J}}$, hence the Gaussian measure $d\nu_{\mathcal{J}}$ is well-defined.

Since the slice assignments, the fields, the measure and the integrand are now factorized over the connected components of $\bar{\mathcal{J}}$, the logarithm of Z is easily computed as exactly the same sum but restricted to two-levels spanning trees:

$$\log Z^{j_{max}}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J} \text{ tree}} \int dw_{\mathcal{J}} \int d\nu_{\mathcal{J}} \partial_{\mathcal{J}} \left[\prod_{\mathcal{B}} \prod_{a \in \mathcal{B}} \left(W_a(\sigma_a, \chi^{\mathcal{B}}, \bar{\chi}^{\mathcal{B}}) \right) \right], \quad (3.8)$$

where the sum is the same but conditioned on $\bar{\mathcal{J}} = \mathcal{F}_B \cup \mathcal{F}_F$ being a *spanning tree* on $\mathcal{W} = [1, \dots, n]$. The main result is the convergence of this representation uniformly in j_{max} for λ in a certain domain, allowing to perform in this domain the ultraviolet limit of the theory. More precisely

Theorem 3.1 *Fix $\rho > 0$ small enough. The series (3.8) is absolutely convergent, uniformly in j_{max} , for λ in the small open cardioid domain Card_{ρ} defined by $|\lambda| < \rho \cos^2[(\text{Arg } \lambda)/2]$ (see Figure 8). Its ultraviolet limit $\log Z(\lambda) = \lim_{j_{max} \rightarrow \infty} \log Z^{j_{max}}(\lambda)$ is therefore well-defined and analytic in that cardioid domain; furthermore it is the Borel sum of its perturbative series in powers of λ .*

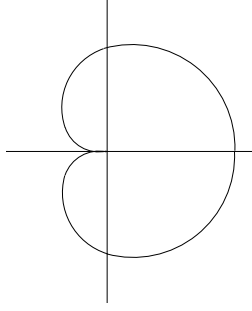


Figure 8: A Cardioid Domain

4 The Bounds

4.1 Grassmann Integrals

Each vertex W_a is developed as a sum over J_a and over connected graphs G_a which contains the marks of J_a exactly once. It is then important to emphasize explicitly the constraints that the scale subsets J_a obey to a *hard core constraint inside each block*. Indeed if there was any non empty such intersection one would integrate twice with respect to a same Grassmann variable corresponding to that block and to the same frequency. Hence without changing the value of the expansion we can factor out a term $\prod_{\mathcal{B}} \prod_{a,b \in \mathcal{B}} \mathbf{1}(J_a \cap J_b = \emptyset)$, where the function $\mathbf{1}(E)$ means the characteristic function of the event E .

The Grassmann Gaussian part of the functional integral (3.8) is then treated as in [12], resulting in a similar computation. Let us for the moment fix the forest \mathcal{F}_F , hence also the two ends a, b of each Fermionic link $\ell_{\mathcal{F}}$, and let us also fix the scale j_{ℓ_F} off each Fermionic link ℓ_F . All these data will be summed later. Definng as in [12] the natural n by n extension of the matrix $Y_{\mathcal{B}, \mathcal{B}'}$ by $\mathbf{Y}_{ab} = Y_{\mathcal{B}(a)\mathcal{B}(b)}(w_{\ell_F})$, we can evaluate

$$\begin{aligned} & \int \left[\prod_{\mathcal{B}} \prod_j (d\bar{\chi}_j^{\mathcal{B}} d\chi_j^{\mathcal{B}}) \right] e^{-\sum_{j=0}^{max} \bar{\chi}_j^{\mathcal{B}} Y_{\mathcal{B}\mathcal{B}'}(w_{\ell_F}) \chi_j^{\mathcal{B}'}} \prod_{\substack{\ell_F \in \mathcal{F}_F \\ \ell_F = (a,b)}} \left(\chi_{j_{\ell_F}}^{\mathcal{B}(a)} \bar{\chi}_{j_{\ell_F}}^{\mathcal{B}(b)} + \chi_{j_{\ell_F}}^{\mathcal{B}(b)} \bar{\chi}_{j_{\ell_F}}^{\mathcal{B}(a)} \right) \\ &= \left(\prod_{\mathcal{B}} \prod_{a,b \in \mathcal{B}} \mathbf{1}(J_a \cap J_b = \emptyset) \right) \left(\mathbf{Y}_{\hat{a}_1 \dots \hat{a}_k}^{\hat{b}_1 \dots \hat{b}_k} + \mathbf{Y}_{\hat{b}_1 \dots \hat{a}_k}^{\hat{a}_1 \dots \hat{b}_k} + \dots + \mathbf{Y}_{\hat{b}_1 \dots \hat{a}_k}^{\hat{a}_1 \dots \hat{b}_k} \right), \end{aligned} \quad (4.1)$$

where $k = |\mathcal{F}|$ and the sum runs over the 2^k ways to exchange the ends a_i and b_i of each ℓ_F , and the Y factors are (up to a sign) the minors of Y with the lines $b_1 \dots b_k$ and the columns $a_1 \dots a_k$ deleted. The most important factor in (4.1) is $\prod_{\mathcal{B}} \prod_{a,b \in \mathcal{B}} \mathbf{1}(J_a \cap J_b = \emptyset)$ which ensures the disjointness of the slices in each block.

Positivity of the Y covariance means as usual that the Y minors are all bounded by 1 [29, 12], namely for any a_1, \dots, a_k and b_1, \dots, b_k ,

$$\left| \mathbf{Y}_{\hat{b}_1 \dots \hat{a}_k}^{\hat{a}_1 \dots \hat{b}_k} \right| \leq 1. \quad (4.2)$$

4.2 Bosonic Integrals

The main problem is now the evaluation of the Bosonic integral $\int d\nu(\sigma)$ in (3.8). Since it factorizes over the Bosonic blocks, it is sufficient to bound separately this integral in each fixed block \mathcal{B} . Consider such a block \mathcal{B} , and the fixed set of slice-subsets and graphs S_a, G_a of that block. We shall define $J(\mathcal{B}) = \cup_{a \in \mathcal{B}} J_a$, hence it is the set of slices present in the block \mathcal{B} (remember the factor $\prod_{a,b \in \mathcal{B}} \mathbf{1}(J_a \cap J_b = \emptyset)$ which ensures that the J_a are all disjoint for $a \in \mathcal{B}$).

In the block, the Bosonic forest \mathcal{F}_B restricts to a Bosonic tree \mathcal{T}_B , and the Bosonic Gaussian measure $d\nu$ restricts to $d\nu_{\mathcal{B}}$ defined by

$$\int d\nu_{\mathcal{B}} F_{\mathcal{B}} = \left[e^{\frac{1}{2} \sum_{a,b \in \mathcal{B}} X_{ab}(w_{\ell_B}) \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} F_{\mathcal{B}}} \right]_{\sigma=0}. \quad (4.3)$$

The Bosonic integrand is obtained by evaluating the action of the coupling derivatives in

$$\begin{aligned} F_{\mathcal{B}} &= \prod_{a \in \mathcal{B}} \left[\prod_{e \in E_{\mathcal{B}}^a} \left(\frac{\partial}{\partial \sigma_e} \right) W_{j_a} \right] \\ &= \prod_{a \in \mathcal{B}} \left[\prod_{e \in E_{\mathcal{B}}^a} \left(\frac{\partial}{\partial \sigma_e} \right) \prod_{j \in J_a} \left[\int_0^1 dt_j e^{V_j(t, \sigma_a)} \right] A_{G_a}(t, \sigma_a) \right] \end{aligned} \quad (4.4)$$

where $E_{\mathcal{B}}^a$ runs over the set of all edges in $\mathcal{T}_{\mathcal{B}}$ which end at vertex a , hence $|E_{\mathcal{B}}^a| = d_a(\mathcal{T}_{\mathcal{B}})$, the degree or coordination of the tree $\mathcal{T}_{\mathcal{B}}$ at vertex a . The derivatives $\prod_{e \in E_{\mathcal{B}}^a}$ act⁸ either on the amplitudes A_{G_a} or derive new loop vertices from the exponential $\prod_{j \in J_a} e^{V_j(\sigma_a)}$.

When \mathcal{B} has more than one vertex, since $\mathcal{T}_{\mathcal{B}}$ is a tree, each vertex $a \in \mathcal{B}$ is touched by at least one derivative. We can evaluate the derivatives in (4.4) through the Faà di Bruno formula:

$$\left(\prod_{e \in E} \frac{\partial}{\partial \sigma_e} \right) f(g(\sigma)) = \sum_{\pi} f^{|\pi|}(g(\sigma)) \prod_{B \in \pi} \left[\left(\prod_{e \in B} \frac{\partial}{\partial \sigma_e} \right) g(\sigma) \right], \quad (4.5)$$

where π runs over the partitions of the set S and B runs through the blocks of the partition π . For the purpose of this paper we won't need to evaluate too precisely the result, but let us remark that

- the exponential $\prod_{j \in J_a} e^{V_j(\sigma_a)}$ cannot disappear since the exponential function is its own derivative,
- the derivatives which act on the graph integrand $A_{G_a}(t, \sigma_a)$ must act on the R resolvent factors and create therefore new propagators sandwiched by resolvents, through $\frac{\partial}{\partial \sigma} R = 2i\sqrt{\lambda}RCR$,
- the derivatives which act on the exponential create new loop vertices of the type $i\sqrt{\lambda}C(R-1)$ or T_j counterterms.

We do not try to compensate eventual tadpoles T_j created by these $\frac{\partial}{\partial \sigma}$ derivatives, since the good factors that have been prepared by the slice testing expansion are more than enough to pay for these uncompensated counterterms.

Hence we can write informally the Bosonic integrand after action of derivatives as

$$\int d\nu_{\mathcal{B}} F_{\mathcal{B}} = \int d\nu_{\mathcal{B}} e^{V(\mathcal{B}, J_a)} \sum_{G_{\mathcal{B}} \in \mathcal{G}(\mathcal{B})} A_{G_{\mathcal{B}}}(\sigma). \quad (4.6)$$

where the graphs in $\mathcal{G}(\mathcal{B})$ are resolvent graphs, but which are no longer minimal, nor connected over all vertices of \mathcal{B} . They have still the same set of marks $J(\mathcal{B})$, but their order in λ is now bounded by $|J(\mathcal{B})| + |\mathcal{B} - 1|$, since the tree of $|\mathcal{B}| - 1$ derivatives connecting \mathcal{B} brings down exactly $|\mathcal{B}| - 1$ factors λ from the exponential and from resolvents.

Furthermore the amplitudes $A_{G_{\mathcal{B}}}(\sigma)$ are given by a formula intermediate between (2.15) and (2.16). There can be up to $2|\mathcal{B}| - 1$ additional (uncompensated) tadpole insertions. Furthermore the resolvent of the additional tadpoles may not be $R - 1$ but can be R 's.

Then we perform a Cauchy-Schwarz inequality with respect to the positive measure $d\nu_{\mathcal{B}}$ to separate the graphs from the remaining interaction:

$$\sum_{G_{\mathcal{B}} \in \mathcal{G}(\mathcal{B})} \left| \int d\nu_{\mathcal{B}} e^{V(\mathcal{B}, J_a)}(\sigma) A_{G_{\mathcal{B}}}(\sigma) \right| \leq \sum_{G_{\mathcal{B}} \in \mathcal{G}(\mathcal{B})} \left(\int d\nu_{\mathcal{B}} \prod_a e^{2|V(\mathcal{B}, J_a)|} \right)^{1/2} \left(\int d\nu_{\mathcal{B}} |A_{G_{\mathcal{B}}}(\sigma)|^2 \right)^{1/2}. \quad (4.7)$$

⁸ When the block \mathcal{B} is reduced to a single vertex a , there is no derivative to compute and the integrand reduces simply to $F_{\mathcal{B}} = \prod_{j \in J_a} \left[\int_0^1 dt_j e^{V_j(t, \sigma_a)} \right] A_{G_a}(t, \sigma_a)$. This case is easy.

4.3 Non-Perturbative Bound I: The Remaining Interaction

In this subsection we bound $\int d\nu_{\mathcal{B}} \prod_a e^{2|V(\mathcal{B}, J_a)|}$ in (4.7).

Lemma 4.1 *For g in the cardioid domain Card_ρ defined in Theorem 3.1 we have*

$$|\exp(\text{Tr} V_j(\sigma))| \leq \exp(0(1) \rho j + |\lambda|^{1/2} \sin(\phi/2) \text{Tr} \sigma + \rho \text{Tr}(C_{\leq j} \sigma C_j \sigma)). \quad (4.8)$$

Proof Using (2.30) we write, putting $\text{Arg} \lambda = \phi$

$$\begin{aligned} |\exp(\text{Tr} V_j(\sigma))| &= \exp \Re \left(\int_0^1 du_j \text{Tr} [6\lambda T_j T_{\leq j}(t) + 2i\sqrt{\lambda} T_j \sigma - i\sqrt{\lambda} [R_{\leq j}(t) - 1] C_j \sigma] \right) \\ &\leq \exp(0(1) (\rho j + |\lambda|^{1/2} \sin(\phi/2) \text{Tr} \sigma + |\lambda| |\text{Tr}(C_j \sigma R_{\leq j} C_{\leq j} \sigma)|)) \\ &\leq \exp(0(1) (\rho j + |\lambda|^{1/2} \sin(\phi/2) \text{Tr} \sigma + \rho \text{Tr}(C_{\leq j}^{1/2} \sigma C_j \sigma C_{\leq j}^{1/2}))) \\ &= \exp(0(1) (\rho j + |\lambda|^{1/2} \sin(\phi/2) \text{Tr} \sigma + \rho \text{Tr}(C_{\leq j} \sigma C_j \sigma))). \end{aligned} \quad (4.9)$$

For the first to second line we used that $(R_{\leq j}(t) - 1)C_j \sigma = -2i\sqrt{\lambda} R_{\leq j} C_{\leq j} \sigma C_j \sigma$. Then for A positive⁹ Hermitian and B bounded we have $|\text{Tr} AB| \leq \|B\| \text{Tr} A$. Indeed if B is diagonalizable with eigenvalues μ_i , computing the trace in a diagonalizing basis we have $|\sum_i A_{ii} \mu_i| \leq \max_i |\mu_i| \sum_i A_{ii}$; if B is not diagonalizable we can use a limit argument. We can now remark that for any Hermitian operator L we have, if $|\text{Arg} \lambda| = |\phi| < \pi$, $\|(1 - i\sqrt{\lambda} L)^{-1}\| \leq \frac{1}{\cos(\phi/2)}$. We can therefore apply these arguments to $A = C_{\leq j}^{1/2} \sigma C_j \sigma C_{\leq j}^{1/2}$ (which is Hermitian positive) and $B = C_{\leq j}^{-1/2} R_{\leq j} C_{\leq j}^{1/2}$. Indeed

$$\|B\| = \|R_{\leq j}\| = \|(1 - i\sqrt{\lambda} C_{\leq j}^{1/2} \sigma C_{\leq j}^{1/2})^{-1}\| \leq \frac{1}{\cos(\phi/2)}. \quad (4.10)$$

We conclude since in the cardioid $\frac{|\lambda|}{\cos^2(\phi/2)} \leq \rho$. \square

We can now bound the first factor in the Cauchy-Schwarz inequality (4.7).

Theorem 4.2 (Bosonic Integration) *For ρ small enough and for any value of the w interpolating parameters*

$$\left(\int d\nu_{\mathcal{B}} e^{2 \sum_{j \in J(\mathcal{B})} |V_j(\sigma_a)|} \right)^{1/2} \leq e^{O(1) \rho \sum_{j \in J(\mathcal{B})} j}. \quad (4.11)$$

Proof Remark that the first term in $0(1) \rho j$ in (4.8) gives precisely a bound in $O(1) \rho \sum_{j \in J(\mathcal{B})} j$. So it remains to check that

$$\int d\nu_{\mathcal{B}} e^{O(1) (|\lambda|^{1/2} \sin(\phi/2) \text{Tr} \sigma + \rho \text{Tr}(C_{\leq j} \sigma C_j \sigma))}^{1/2} \leq e^{O(1) \rho \sum_{j \in J(\mathcal{B})} j}. \quad (4.12)$$

Applying Lemma 4.1 we get

$$\int d\nu_{\mathcal{B}} \prod_{a \in \mathcal{B}} e^{2 \sum_{j \in J_a} |V_j(\sigma_a)|} \leq \int d\nu_{\mathcal{B}} e^{\frac{1}{2} \langle \sigma, \mathbf{Q} \sigma \rangle + \langle \sigma, \mathbf{P} \rangle} \quad (4.13)$$

where \mathbf{Q} is a symmetric positive matrix in the big vector space \mathbf{V} which is the tensor product of the spatial space $L_2([0, 1]^2)$ with the “replica space” generated by the orthonormal basis $\{e_a\}$, $a \in \mathcal{B}$. \mathbf{P} is a constant function in $L_2([0, 1]^2)$, which takes the (single) value $P = O(1) |\lambda|^{1/2} \sin(\phi/2)$.

More precisely \mathbf{Q} is diagonal in replica space and \mathbf{Q} and \mathbf{P} are defined by the equations

$$\begin{aligned} \langle \sigma, \mathbf{Q} \sigma \rangle &= \sum_{a \in \mathcal{B}} \sum_{j \in J_a} \langle \sigma_a, Q_j \sigma_a \rangle, \quad \langle \sigma_a, Q_j \sigma_a \rangle \equiv O(1) \rho \text{Tr}(C_{\leq j} \sigma_a C_j \sigma_a), \\ \langle \sigma, \mathbf{P} \rangle &= \sum_{a \in \mathcal{B}} \sum_{j \in J_a} P \text{Tr} \sigma_a = P \sum_{a \in \mathcal{B}} |J_a| \text{Tr} \sigma_a. \end{aligned} \quad (4.14)$$

⁹We usually simply say positive for “non-negative”, i. e. each eigenvalue is strictly positive or zero.

Each Q_j is positive and using the bounds (2.5) it is easy to check that the kernel of Q_j is bounded by

$$Q_j(x, y) \leq O(1)\rho j e^{-M^j|x-y|}. \quad (4.15)$$

Hence, since we work on a fixed square of unit volume in \mathbb{R}^2 the following lemma follows easily.

Lemma 4.3 *Uniformly in j_{max}*

$$\text{Tr } Q_j \leq O(1)\rho j, \quad (4.16)$$

$$\|Q_j\| \leq O(1)\rho M^{-j/2}. \quad (4.17)$$

The second bound (4.17) is absolutely not optimal but enough for what is needed below. It just comes from bounding first the norm of $e^{-M^j|x-y|}$ by its Hilbert-Schmidt norm which is proportional to M^{-j} (we are in two dimensions) and then bounding the j factor in (4.15) by $M^{j/2}$.

The covariance \mathbf{X} of the Gaussian measure $d\nu_{\mathcal{B}}$ is a symmetric matrix on the big space \mathbf{V} , which is the tensor product of the identity in space times the matrix $X_{ab}(w_{\ell_B})$ in the replica space. Defining $\mathbf{A} \equiv \mathbf{X}\mathbf{Q}$, we have

Lemma 4.4 *The following bounds hold uniformly in j_{max}*

$$\text{Tr } \mathbf{A} \leq O(1)\rho \sum_{j \in J(\mathcal{B})} j, \quad (4.18)$$

$$\|\mathbf{A}\| \leq O(1)\rho. \quad (4.19)$$

Proof Since $\mathbf{Q} = \sum_{a \in \mathcal{B}} \sum_{j \in J_a} \mathbf{Q}_j$ is diagonal in replica space we find that

$$\text{Tr } \mathbf{A} = \sum_{a \in \mathcal{B}} \sum_{j \in J_a} \text{Tr } \mathbf{X}\mathbf{Q}_j = \sum_{a \in \mathcal{B}} \sum_{j \in J_a} X_{aa}(w_{\ell_B}) \text{Tr } Q_j = \sum_{a \in \mathcal{B}} \sum_{j \in J_a} \text{Tr } Q_j \leq O(1)\rho \sum_{j \in J(\mathcal{B})} j. \quad (4.20)$$

where in the last inequality we used (4.16). Furthermore by the triangular inequality in (4.14) and using (4.17)

$$\|\mathbf{A}\| \leq \sum_{a \in \mathcal{B}} X_{aa}(w_{\ell_B}) \sum_{j \in J_a} \|Q_j\| = \sum_{a \in \mathcal{B}} \sum_{j \in J_a} O(1)\rho M^{-j/2} \leq O(1)\rho. \quad (4.21)$$

where we used the fundamental fact that all vertices $a \in \mathcal{B}$ have *disjoint subsets of scales* J_a . \square

We can now complete the proof of Theorem 4.2. Since $\text{Tr } \mathbf{A}^n \leq \text{Tr } \mathbf{A} \|\mathbf{A}\|^n$, by (4.19) for ρ small enough the series $\sum_{n=1}^{\infty} \text{Tr } \mathbf{A}^n$ converges and is bounded by $2 \text{Tr } \mathbf{A}$. This justifies the computation

$$\begin{aligned} \int d\nu_{\mathcal{B}} e^{\frac{1}{2}\langle \sigma, \mathbf{Q}\sigma \rangle + \langle \sigma, \mathbf{P} \rangle} &= e^{\frac{1}{2}\langle \mathbf{P}, \mathbf{X}(1-\mathbf{A})^{-1}\mathbf{P} \rangle} [\det(1-\mathbf{A})]^{-1/2} \\ [\det(1-\mathbf{A})]^{-1/2} &= e^{\frac{1}{2} \sum_{n=1}^{\infty} (\text{Tr } \mathbf{A}^n)/n} \leq e^{\text{Tr } \mathbf{A}} \leq e^{O(1)\rho \sum_{j \in J(\mathcal{B})} j}. \end{aligned} \quad (4.22)$$

Moreover

$$\begin{aligned} e^{\frac{1}{2}\langle \mathbf{P}, \mathbf{X}(1-\mathbf{A})^{-1}\mathbf{P} \rangle} &\leq e^{\frac{1}{2}\|(1-\mathbf{A})^{-1}\| \langle \mathbf{P}, \mathbf{X}\mathbf{P} \rangle} \leq e^{O(1)\rho \sum_{ab} |J_a| X_{ab}(w_{\ell_B}) |J_b|} \\ &\leq e^{O(1)\rho \sum_{a \in \mathcal{B}, b \in \mathcal{B}} |J_a| \cdot |J_b|} = e^{O(1)\rho |J(\mathcal{B})|^2} \leq e^{O(1)\rho \sum_{j \in J(\mathcal{B})} j}. \end{aligned} \quad (4.23)$$

where we used that $X_{ab}(w_{\ell_B}) \leq 1$ for any $\{w_{\ell_B}\}$ and for the last inequality we used again that vertices $a \in \mathcal{B}$ have disjoint subsets of scales J_a .

This completes the proof of Theorem 4.2. \square

4.4 Non-Perturbative Bounds II: Getting Rid of Resolvents

In this subsection we now explain how to bound, for a fixed $G_{\mathcal{B}}$, the second factor

$$I = \left(\int d\nu_{\mathcal{B}}(\sigma) |A_{G_{\mathcal{B}}}(\sigma)|^2 \right)^{1/2} \quad (4.24)$$

of the Cauchy-Schwarz inequality (4.7). We shall not try to establish sharp bounds on I , just bounds sufficient for the proof of Theorem 3.1. This is still a non-perturbative problem, since the resolvents in the $A_{G_{\mathcal{B}}}(\sigma)$, if expanded in power series of σ and integrated out with respect to $d\nu_{\mathcal{B}}$, would lead to infinite divergent series of Feynman graphs. Hence we shall use the norm bound (4.10) to get rid of these resolvents.

For this we write first

$$I^2 = \int d\nu_{\mathcal{B}} A_{G_{\mathcal{B}}}(\sigma) A_{\bar{G}_{\mathcal{B}}}(\sigma) \quad (4.25)$$

where $\bar{G}_{\mathcal{B}}(\sigma)$ is the complex conjugate graph, made with complex conjugate resolvents. We consider $G_{\mathcal{B}} \cup \bar{G}_{\mathcal{B}}$ as a (not connected) graph with twice the number of propagators and vertices of $G_{\mathcal{B}}$.

Let us call $n \geq 2$ the order of perturbation theory for $G_{\mathcal{B}}$ (i.e. the number of its σ propagators). The order of perturbation for $G_{\mathcal{B}} \cup \bar{G}_{\mathcal{B}}$ is $2n$, and the number of its c-propagators is $4n$. Ordering the slices of $J(\mathcal{B})$ as $j_1 < j_2 < \dots < j_p$, we have $p = |J(\mathcal{B})| \geq n/2$ marked c-propagators, one for each slice j_k , $k = 1, \dots, p$. Indeed $G_{\mathcal{B}}$ came from a minimal resolvent graph $G \in \mathcal{G}$ for a certain set of slices $J(\mathcal{B})$, and the definition of minimality implied that the order of G was at most $|J(\mathcal{B})|$; moreover $G_{\mathcal{B}}$ was obtained from G by adding at most $|\mathcal{B}| - 1$ further σ propagators through the MLVE (see subsection 4.2).

We need now to better explicit the fact that the marked c-propagators cannot be tadpoles. This is indeed the source of the good factors which make the expansion converge. For this we simply expand the $R - 1$ factors of the tadpoles of the graph G as $\pm 2i\sqrt{\lambda}RC\sigma$. Then we contract all the σ fields produced in this way. Each Wick contraction step decreases the number of these fields by two if they contract between themselves and by one if they contract to R factors, hence in any such Wick contraction the number of such σ 's decreases at least by one. At the end of this step we obtain a set of new resolvent graphs which have the following properties:

- they have a set M of marked c-propagators which are now explicitly not tadpoles, with indices $j_1 < j_2 < \dots < j_p$, with $p \geq n/2$; moreover all these marked propagators have value C_{j_k} , not RC_{j_k} . We say that they have been cleaned from their resolvent factors.
- their total order of perturbation theory (number of σ -propagators) is at most $4n$.
- the not marked c-propagators have factors either C or RC .

The amplitudes for these graphs are given by (2.15) hence still include a resolvent R for typically many c-propagators.

Our next step is to get rid of all these R factors, essentially using the fact that their norm is bounded by $\cos^{-1}(\phi/2)$ in the cardioid domain. This is not trivial and we shall rely on the technique of recursive Cauchy-Schwarz (CS) inequalities of [20, 21]. Consider a fixed connected intermediate field graph G which is a connected component of the previous list, with order $m \leq 4n$. It has $2m$ c-propagators. The key definition to define the CS inequality is that of a balanced cut. Since it is recursive, we need first to consider a slightly more general class of resolvent amplitudes, for intermediate field vacuum graphs G which have a particular set R of their c-propagators called resolvent propagators, since they bear resolvents. The other c-propagators, bearing no resolvents, are called *cleaned*. The amplitudes of such partly cleaned graphs are given by the following formula which generalizes (2.15)

$$A_{G,R}(t, \sigma) = \prod_{v \in V(G)} [(-\lambda) \int_{[0,1]^2} d^2 x_v] \prod_{\ell \in R} [R(\sigma) C_{j(\ell)}](x_\ell, x'_\ell) \prod_{\ell \notin R} [C_{j(\ell)}](x_\ell, x'_\ell). \quad (4.26)$$

Hence $R = CP(G)$, the full set of c-propagators of G , correspond to the resolvent amplitudes (2.15), and $R = \emptyset$ correspond to the ordinary perturbative Feynman amplitudes for intermediate field graphs. This definition generalizes in a straightforward way to the case where c-propagators have certain slice restrictions.

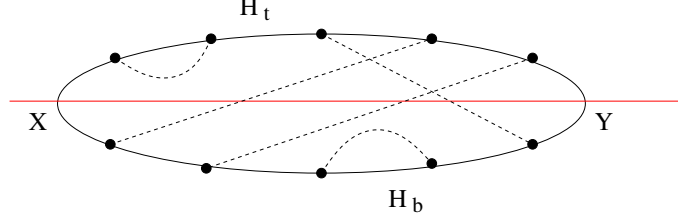


Figure 9: A resolvent graph with a balanced cut. Here σ -propagators are pictured as dotted lines.

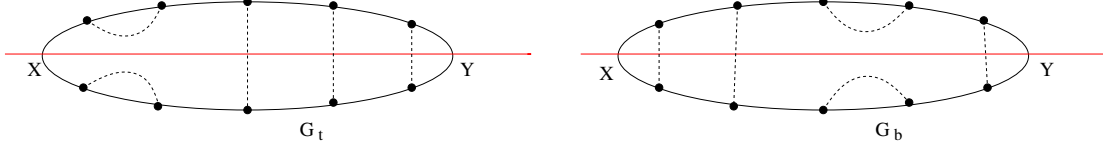


Figure 10: The top and bottom mirror graphs for the balanced cut

We define a *balanced $X - Y$ cut* for (G, R) as a partition of the graph G into two pieces which we call the top and bottom chains, H_t and H_b , each containing the *same number of resolvent propagators* (up to one unit if the initial number of resolvent propagators is odd). H_t and H_b are each made of a chain of p or $p - 1$ resolvent propagators, plus the two half resolvent propagators X and Y at the ends of the chain, plus an arbitrary number of cleaned propagators (it needs not be the same number in H_t and H_b). To these two chains are hooked the same number q of half σ -propagators which cross the cut, plus inner σ -propagators which do not cross the cut. Remark that σ propagators have no reason to occur at symmetric positions along the top and bottom chains (see Figure 9).

Balanced cuts for an intermediate field connected vacuum graph (G, R) with $R \neq \emptyset$ can be obtained in many different ways. A nice way to define such cuts is to first select a spanning tree of σ propagators of G . Then turning around the tree provides a well defined cyclic ordering of the $2m$ c-propagators of the graph (jumping over the σ -propagators not in the tree). Balanced cuts are then obtained by first *contracting all cleaned propagators along the cycle*, and then selecting an antipodal¹⁰ pair (X, Y) among the resolvent propagators left in that cycle. We then cut the cycle across that pair (see Figure 9).

To any such balanced cut is associated a Cauchy-Schwarz (CS) inequality. It bounds the resolvent amplitude A_G (see (2.15)) by the geometric mean of the amplitudes of the two graphs $G_t = H_t \cup \bar{H}_t$ and $G_b = H_b \cup \bar{H}_b$. These two graphs are obtained by gluing H_t and H_b with their mirror image along the cut. Remark that in this gluing the σ propagators crossing the cut are fully disentangled: in G_t and G_b they no longer cross each other, see Figure 10. Remark also that the right hand side of the CS inequality, hence the bound obtained for A_G , is a priori different for different balanced cuts.

In such a CS inequality, something crucial can be gained. In the cardioid we recall that by (4.10) $\|R\| \leq \cos^{-1}(\phi/2)$.

Now the two propagators X and Y crossed by the balanced cut at the end of the top and bottom chain, which had values RC in the amplitude A_G , can be replaced by two *ordinary* propagators C in the amplitudes of G_t and G_b , loosing simply a factor $\cos^{-2}(\phi/2)$ for the two norms of R . Hence they are cleaned.

Lemma 4.5 *For any balanced $X - Y$ cut*

$$|A_{G,R}(\sigma)| \leq \frac{1}{\cos^2(\phi/2)} \sqrt{A_{G_t,R-\{X \cup Y\}}(\sigma)} \sqrt{A_{G_b,R-\{X \cup Y\}}(\sigma)} \quad (4.27)$$

uniformly in σ . Hence we have cleaned the two resolvent propagators X and Y crossed by the cut.

Proof Each CS inequality is simply obtained by writing

$$\langle H_t, \mathcal{O} H_b \rangle \leq \|\mathcal{O}\| \sqrt{\langle H_t H_t \rangle} \sqrt{\langle H_b H_b \rangle} \quad (4.28)$$

¹⁰Or almost antipodal if the number of resolvents is not even; this can happen only at the first CS step.

in the tensor product of $2+q$ Hilbert spaces corresponding to the two end c-propagators and the q crossing σ -propagators. We symmetrize first the operators RC of the two cut propagators, writing them as $C^{1/2}BC^{1/2}$ with $B = C^{-1/2}RC^{1/2}$. The operator $\mathcal{O} = B \otimes \Pi \otimes B$ in (4.28) is the tensor product of the two end operators B and of a permutation operator Π for the remaining $H^{\otimes q}$ tensor product of the q crossing σ -propagators. Therefore $\|\mathcal{O}\| \leq \|B\|^2 \|\Pi\|$. Any permutation operators has eigenvalues which are roots of unity, hence has norm bounded by 1, and $\|B\| = \|R\| \leq \cos^{-1}(\phi/2)$. \square

Again remark that this lemma can also be applied to the case where the c-propagators have certain slice restrictions, in which case these restrictions are carried to their copies in G_t and G_b .

Starting with a full resolvent graph, the inductive CS inequalities of [20, 21] consist in iterating lemma 4.5 until no resolvents are left anywhere. In this way we can therefore reach a bound made of a geometric mean of 2^m ordinary perturbative amplitudes for an initial resolvent graph of order m . To understand the result of the induction, let us observe that

- Only at the first step the number of resolvent propagators can be odd. In that case we choose an almost antipodal pair (antipodal up to half a unit): but at all later stages the mirror gluing creates an even number of resolvents and we can choose truly antipodal pairs.
- The result of m complete inductive layers of CS steps applied to a starting graph G of order m is a family $F_m^{\mathcal{C}}(G)$ of 2^m graphs, which depends on the inductive choices of all the balanced cuts of the induction. The set of these choices is noted \mathcal{C} . The graphs of $F_q^{\mathcal{C}}(G)$ are called q -th layer graphs and can be pictured to stand at the leaves of a rooted binary tree, with the initial graph G standing at the root. \mathcal{C} is a choice of a balanced cut for each vertex of that rooted binary tree. It splits the parent graph G' of layer $q-1$ for the parent edge of the vertex into two children graphs G'_t and G'_b of layer q , one for each of the two children-edges of the vertex.
- Although the graphs in the family $F_q^{\mathcal{C}}(G)$ may have very different orders, they all have the same number of resolvents (up to one at most, if the initial number of resolvents was odd).
- No matter which inductive choice \mathcal{C} is made, every c-propagator ℓ of the initial graph G gets finally copied into exactly 2^m c-propagators in the union of all graphs of $F_m^{\mathcal{C}}(G)$. Notice that all these copies have the same slice-attribution $j(\ell)$ than the initial propagator. But they are not at all evenly distributed among the members of the family,

This is summarized in the following lemma.

Lemma 4.6 *For any choice \mathcal{C} of m recursive cuts*

$$|A_G(\sigma)| \leq \left[\prod_{G' \in F_m^{\mathcal{C}}(G)} |A_{G'}| \right]^{2^{-m}} \quad (4.29)$$

uniformly in σ . The amplitudes $A_{G'}$ are computed with coupling constants ρ instead of $|\lambda|$.

Proof Straightforward induction using lemma 4.5. We bound all the factors $\cos^{-2}(\phi/2)$ generated by the CS inequalities by changing the factor $|\lambda^{V(G')}|$ into $|\rho^{V(G')}|$. Indeed for each pair of resolvents destroyed by a CS inequality there is an independent coupling constant factor $|\lambda|$, and in the cardioid we have $|\lambda| \cos^{-2}(\phi/2) \leq \rho$. \square

Recall that the amplitudes $A_{G'}$ have no resolvent factors any more, hence are ordinary perturbative amplitudes no longer depending on σ . We can now reap the good factors due to the marked propagators and use them to bound all remaining amplitude factors and combinatorics.

4.5 Perturbative Bounds, Combinatorics and Final Bound

We shall be brief, as this section does not contain any new idea. Summarizing the results of the previous section we obtained a geometric average over amplitudes for ordinary graphs G' (not necessarily connected) each with order at most $4n$, and at least $n/2$ different marked slice propagators $j_1 < j_2 < \dots < j_p$, each of which is not a tadpole. The sum over slices for all other propagators of the graph are restricted to $J(\mathcal{B}) = \{j_1, \dots, j_p\}$ (since all slices with indices not in $J(\mathcal{B})$ were deleted when their parameter t_j was put to 0 in (2.28)).

Lemma 4.7 *The amplitude for any such graph G is bounded by*

$$|A_G| \leq O(1)^n \rho^n p^{8n} M^{-2 \sum_{i=1}^p j_i/7} \quad (4.30)$$

Proof We bound the propagators according to (2.5). Consider the highest marked propagator, of scale j_p . Since it is not a tadpole we gain an integration factor M^{-2j} for the integration of one of the two vertices hooked to it with respect to the other. Then we cross the (at most 6) propagators which touch that propagator and consider the next highest uncrossed marked propagator. Iterating, we gain a factor at least $M^{-2 \sum_{i=1}^p j_i/7}$ (with little extra care we could have improved this bound to $M^{-\sum_{i=1}^p j_i/2}$). All other vertex integrations can be bounded by 1 and each sum over slice attributions can be bounded by p . Since there are at most $8n$ such sums we obtain (4.30). \square

Taking ρ small enough, with a fraction of the factor $M^{-2 \sum_{i=1}^p j_i/7}$ in (4.30) we can bound the factor $e^{O(1)\rho \sum_{j \in \mathcal{J}(\mathcal{B})} j}$ in (4.11).

The sum over all combinatorial structures of Feynman graphs at order n and the sum over the forests of section 3 are all similarly bounded by $O(1)^n n^{4n}$. We refer for more details on exact values for such combinatorial factors to [12]. Since $p \geq n/2$ and our graphs have order at most $4n$ we have

$$O(1)^n n^{4n} M^{-O(1)p^2} \leq [O(1)\rho]^n. \quad (4.31)$$

Finally let us describe how to bound the additional factors due to the auxiliary expansion

- the sum over partitions Π is the Bell number B_p , itself bounded by $p!$
- the sum over trees \mathcal{T} is bounded by p^{p-2} by Cayley's theorem
- the w integrals and factors are all bounded by 1
- the crossed counterterms or tadpoles are bound by p^{p-1}
- the sum over derivatives actions ∂w , hence over the modified graphs is certainly bounded by p^{4p}

In conclusion all auxiliary expansion effects factorize over the blocks and in each block of size p are bounded by $p^{O(1)p}$, hence easily beaten by another small fraction of the $M^{-O(1)p^2}$ convergence factor due to the marked renormalized propagators.

Hence we obtain a uniformly convergent bound for the series (3.8) in the cardioid. This achieves the proof of Theorem 3.1.

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